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STATIONARY MEASURES FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS*

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Abstract. Three methods for studying stationary measures of stochastic differential equations with jumps are considered. These stationary measures are given by Markov measures, solutions of Fokker-Planck equations, and long time limits for the distributions of system states.

1. Introduction

Stationary measures for stochastic differential equations (SDEs) and invariant measures for Markov processes have been studied extensively, as in [2, 3, 5, 13, 17], among others. Moreover, there are interesting relationships between stationary measures and invariant measures, such as the one-to-one correspondence between the set of invariant Markov measures (c.f. Definition 3.1 in Section 3) and the set of stationary measures (c.f. [3, 5]), as well as the correspondence between the solutions of Fokker-Planck equations and stationary measures (c.f.[13]). These results play an important role in the development of the theory for random dynamical systems associated with SDEs.

SDEs with jumps and random dynamical systems associated with them are considered by a number of authors (c.f. [1] [2] [7] [10] [11] [15] [17]). We note that Albeverio-Rüdiger-Wu [1] discussed stationary measures for SDEs with jumps, in the context of Lévy type operators and considered mainly infinitesimal invariant measures. The concept of infinitesimal invariant measures is weaker than that of the usual stationary measures. Moreover, Zabczyk [17] studied stationary measures for *linear* SDEs with jumps.

We ask, naturally, whether the above correspondences for usual SDEs also hold for SDEs with jumps. This question will be answered in Sections 3 and 4 in the present paper.

Now we briefly sketch our method. We begin from SDEs with jumps as random dynamical systems and examine their Markov measures. Then we consider SDEs which are driven by Brownian motions and by α-stable processes. By a functional analysis

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technique, stationary measures are investigated. When the coefficients of the SDEs are sufficiently regular, the long time limits for the distributions of the solutions are shown to be stationary measures, with the help of Malliavin analysis.

This paper is arranged as follows. In Section 2, we introduce random dynamical systems and related concepts. Sobolev spaces and α-stable processes are also introduced. The content to obtain stationary measures from Markov measures is in Section 3. In Section 4, we consider special stochastic differential equations driven by Brownian motions and α-stable processes. In Section 5, we study SDEs with jumps, under certain regular conditions on the coefficients.

The following convention will be used throughout the paper: C with or without indices will denote different positive constants (depending on the indices) whose values may change from one place to another.

2. Preliminary

2.1. **Random dynamical systems and related concepts.** Let $D(\mathbb{R}_+,\mathbb{R}^d)$ be the set of all functions which are càdlàg, defined on $\mathbb{R}_+$ and taking values in $\mathbb{R}^d$. We take sample space $\Omega=D(\mathbb{R}_+,\mathbb{R}^d)$. This sample space becomes a complete and separable metric space (c.f.[6]), when endowed with the following Skorohod metric d:

$$d(x,y):=\inf_{\lambda\in\Lambda}\left\{\sup_{s\neq t}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right|+\sum_{m=1}^{\infty}\frac{1}{2^m}\big(1\wedge d_m^{\circ}(x^m,y^m)\big)\right\}$$

for all $x,y\in\Omega$, where $x^m(t):=g_m(t)x(t)$, $y^m(t):=g_m(t)y(t)$ with

$$g_m(t):=\begin{cases}1,\text{if } t\leqslant m,\\ m+1-t,\text{if } m<t<m+1,\\ 0,\text{if } t\geqslant m+1,\end{cases}$$

and

$$d_m^{\circ}(x,y):=\sup_{0\leqslant t\leqslant m}|x(t)-y(\lambda(t))|.$$

Moreover Λ denotes the set of strictly increasing and continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, and $a\wedge b:=\min\{a,b\}$. We identify a càdlàg function $\omega(t)$ with a (canonical) sample ω in the sample space Ω.

The Borel σ-algebra in the sample space Ω under the topology induced by the Skorohod metric d is denoted as $\mathcal{F}$. Note that $\mathcal{F}=\sigma(\omega(t),t\in\mathbb{R}_+)$ (c.f.[6]). Let P be the unique probability measure which makes a canonical process a Lévy process, and denotes $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ the complete natural filtration with respect to P. Thus $(\Omega,\mathcal{F},P;(\mathcal{F}_t)_{t\in\mathbb{R}_+})$ is a complete filtered probability space.

Define for each $t\in\mathbb{R}_+$,

$$(\theta_t\omega)(\cdot)=\omega(t+\cdot)-\omega(t),\quad \omega\in\Omega.$$

Then θ is a one-parameter semigroup on Ω, Ω is invariant with respect to θ, i.e.

$$\theta_t^{-1}\Omega=\Omega,\ \text{for all } t\in\mathbb{R}_+,$$

and P is θ-invariant, i.e.

$$P(\theta_t^{-1}(B))=P(B),\ \text{for all } B\in\mathcal{F},t\in\mathbb{R}_+.$$

Thus $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}_+})$ is a metric dynamical system (DS), as defined in [3]. This metric DS is in fact ergodic, i.e. all measurable θ-invariant sets have probability either 0 or 1. (c.f.[3])

Definition 2.1. *A measurable random dynamical system on the measurable space* $(\mathbb{X}, \mathcal{B})$ *over a metric DS* $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}_+})$ *with time* $\mathbb{R}_+$ *is a mapping*

$$\varphi : \mathbb{R}_+ \times \Omega \times \mathbb{X} \mapsto \mathbb{X}, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$
$$\varphi(t, \omega) := \varphi(t, \omega, \cdot) : \mathbb{X} \mapsto \mathbb{X},$$

such that

(i) Measurability: φ *is* $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{B}/\mathcal{B}$*-measurable, where* $\mathcal{B}(\mathbb{R}_+)$ *is Borel* σ*-algebra of* $\mathbb{R}_+$.

(ii) Càdlàg cocycle property: $\varphi(t, \omega)$ *forms a (perfect) càdlàg cocycle over* θ *if it is càdlàg in* t *and satisfies*

$$\varphi(0, \omega) = id_{\mathbb{X}}, \text{ for all } \omega \in \Omega, \tag{1}$$
$$\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega), \tag{2}$$

for all $s, t \in \mathbb{R}_+$ *and* $\omega \in \Omega$.

A random dynamical system(RDS) induces a skew product flow of measurable maps

$$\begin{aligned} \Theta_t : \Omega \times \mathbb{X} &\mapsto \Omega \times \mathbb{X} \\ (\omega, x) &\mapsto \big(\theta_t\omega, \varphi(t, \omega)x\big). \end{aligned}$$

The flow property $\Theta_{t+s} = \Theta_t \circ \Theta_s$ follows from (2). Denote by $\mathscr{P}(\Omega \times \mathbb{X})$ the probability measures on $(\Omega\times\mathbb{X}, \mathcal{F}\otimes\mathcal{B})$. Moreover, Θ_t acts on $\mu \in \mathscr{P}(\Omega\times\mathbb{X})$ by $(\Theta_t\mu)(C) = \mu(\Theta_t^{-1}C)$, for $C \in \mathcal{F} \otimes \mathcal{B}$, $t \in \mathbb{R}_+$.

Definition 2.2. *A probability measure* $\mu \in \mathscr{P}(\Omega \times \mathbb{X})$ *is called invariant for the skew product flow* Θ_t *if*

(i) the marginal of μ *on* Ω *is* P,

(ii) $\Theta_t\mu = \mu$ *for all* $t \in \mathbb{R}_+$.

If $\mathbb{X}$ is a Polish space with its Borel σ-algebra $\mathcal{B}(\mathbb{X})$, every measure $\mu \in \mathscr{P}(\Omega\times\mathbb{X})$ with marginal P can be uniquely characterized through its factorization

$$\mu(\mathrm{d}\omega, \mathrm{d}x) = \mu_\omega(\mathrm{d}x)P(\mathrm{d}\omega),$$

where $\mu_\omega(\mathrm{d}x)$ is a probability kernel, i.e. for any $B \in \mathcal{B}(\mathbb{X})$, $\mu_\cdot(B)$ is $\mathcal{F}$-measurable; for $P.a.s.\omega \in \Omega$, $\mu_\omega(\cdot)$ is a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$(c.f.[3, p.23]). Thus μ is invariant if and only if

$$\mathbb{E}[\varphi(t, \omega)\mu_\cdot|\theta_t^{-1}\mathcal{F}](\omega) = \mu_{\theta_t\omega}, \quad P.a.s, \tag{3}$$

for all $t \in \mathbb{R}_+$.

2.2. **Sobolev spaces.** Let $\mathcal{C}_0(\mathbb{R}^d)$ be the space of continuous functions f on $\mathbb{R}^d$ satisfying $\lim\limits_{|x|\to\infty} f(x) = 0$ with norm $\|f\|_{\mathcal{C}_0(\mathbb{R}^d)} = \sup\limits_x |f(x)|$. Let $\mathcal{C}_0^2(\mathbb{R}^d)$ be the set of $f \in \mathcal{C}_0(\mathbb{R}^d)$ such that f is 2 times differentiable and the partial derivatives of f with order $\leqslant 2$ belong to $\mathcal{C}_0(\mathbb{R}^d)$. Let $\mathcal{C}_c^n(\mathbb{R}^d)$ stand for the space of all n times differentiable functions on $\mathbb{R}^d$ with compact supports. Let $S(\mathbb{R}^d)$ be the Schwartz space of all rapidly decreasing real valued $\mathcal{C}^\infty$ functions on $\mathbb{R}^d$ and $S'(\mathbb{R}^d)$ the space of all tempered distributions on $\mathbb{R}^d$. Let

$\hat{f}$ and $\check{f}$ denote the Fourier transform and the Fourier inversion transform of $f \in S(\mathbb{R}^d)$, respectively, i.e.

$$\hat{f}(u) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\langle u,x\rangle} f(x)\mathrm{d}x,$$
$$\check{f}(u) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\langle u,x\rangle} f(x)\mathrm{d}x,$$

for all $u \in \mathbb{R}^d$. And Fourier transforms and the Fourier inversion transforms can be defined on $S'(\mathbb{R}^d)$ by the same means to the above one. We introduce the following Sobolev space

$$\mathbb{H}^{\lambda,2}(\mathbb{R}^d) := \{f \in S'(\mathbb{R}^d) : \|f\|_{\lambda,2} < \infty\},$$

for any $\lambda \in \mathbb{R}$, where

$$\|f\|^2_{\mathbb{H}^{\lambda,2}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1+|u|^2)^{\lambda} |\hat{f}(u)|^2 \mathrm{d}u.$$

In particular, $\mathbb{H}^{0,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.

2.3. **Rotation invariant α-stable process.**

Definition 2.3. *A process $L = (L_t)_{t\geqslant 0}$ with $L_0 = 0$ a.s. is a d-dimensional Lévy process if*

(i) L has independent increments; that is, $L_t - L_s$ is independent of $L_v - L_u$ if $(u,v) \cap (s,t) = \emptyset$;

(ii) L has stationary increments; that is, $L_t - L_s$ has the same distribution as $L_v - L_u$ if $t - s = v - u > 0$;

(iii) L_t is stochastically continuous;

(iv) L_t is right continuous with left limit.

Its characteristic function is given by

$$\mathbb{E}\left(\exp\{i\langle z, L_t\rangle\}\right) = \exp\{t\Psi(z)\}, \quad z \in \mathbb{R}^d.$$

The function $\Psi : \mathbb{R}^d \to \mathcal{C}$ is called the characteristic exponent of the Lévy process L. By Lévy-Khintchine formula, there exist a nonnegative-definite $d \times d$ matrix Q, a measure ν on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d\setminus\{0\}} (|u|^2 \wedge 1)\nu(\mathrm{d}u) < \infty,$$

and $\gamma \in \mathbb{R}^d$ such that

$$\begin{aligned} \Psi(z) &= i\langle z,\gamma\rangle - \frac{1}{2}\langle z, Qz\rangle \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\langle z,u\rangle} - 1 - i\langle z,u\rangle 1_{|u|\leqslant 1}\right)\nu(\mathrm{d}u). \end{aligned} \tag{4}$$

The measure ν is called the Lévy measure.

Definition 2.4. *For $\alpha \in (0,2)$. A d-dimensional rotation invariant α-stable process L is a Lévy process such that its characteristic exponent Ψ is given by*

$$\Psi(z) = -C|z|^{\alpha}, \quad z \in \mathbb{R}^d.$$

Thus, for d-dimensional rotation invariant α-stable process L, Lévy measure ν is given by

$$\nu(\mathrm{d}u) = \frac{C_{d,\alpha}}{|u|^{d+\alpha}}\mathrm{d}u,$$

and $Q = 0$ in (4). Moreover,

$$-C|z|^\alpha = \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\langle z,u\rangle} - 1 - i\langle z,u\rangle 1_{|u|\leqslant 1}\right)\frac{C_{d,\alpha}}{|u|^{d+\alpha}}\mathrm{d}u.$$

Define

$$(\mathcal{L}_\alpha f)(x) := \int_{\mathbb{R}^d\setminus\{0\}} \left(f(x+u) - f(x) - \langle \partial_x f(x), u\rangle 1_{|u|\leqslant 1}\right)\frac{C_{d,\alpha}}{|u|^{d+\alpha}}\mathrm{d}u$$

on $\mathcal{C}_0^2(\mathbb{R}^d)$. And then for $\xi \in \mathbb{R}^d$

$$(\mathcal{L}_\alpha e^{i\langle\cdot,\xi\rangle})(x) = e^{i\langle x,\xi\rangle}\int_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\langle u,\xi\rangle} - 1 - i\langle \xi,u\rangle 1_{|u|\leqslant 1}\right)\frac{C_{d,\alpha}}{|u|^{d+\alpha}}\mathrm{d}u.$$

By Courrège's second theorem (c.f.[2, Theorem 3.5.5, p.183]), for every $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}
(\mathcal{L}_\alpha f)(x) &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\left[e^{-i\langle x,z\rangle}(\mathcal{L}_\alpha e^{i\langle\cdot,z\rangle})(x)\right]\hat{f}(z)\mathrm{d}z \\
&= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\left[\int_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\langle u,z\rangle} - 1 - i\langle z,u\rangle 1_{|u|\leqslant 1}\right)\frac{C_{d,\alpha}}{|u|^{d+\alpha}}\mathrm{d}u\right]\hat{f}(z)\mathrm{d}z \\
&= -\frac{C}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{i\langle z,x\rangle}|z|^\alpha\hat{f}(z)\mathrm{d}z \\
&= C\cdot[-(-\Delta)^{\alpha/2}f](x).
\end{aligned}$$

Moreover, the following result is well-known (c.f.[1]).

Theorem 2.5. *Let $\mathcal{L}_\alpha$ be as above for $\alpha \in (0,2)$ and $\mathcal{L}_2 = \Delta$, as defined on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$. Then $\mathcal{L}_\alpha$, $0 < \alpha \leqslant 2$, has a unique closed extensions to self-adjoint negative operators on the domain $\mathbb{H}^{\alpha,2}(\mathbb{R}^d)$.*

3. From Markov measures to stationary measures

Let $(\mathbb{U},\mathcal{U},n)$ be a σ-finite measurable space. Let $\{W(t)\}_{t\geqslant 0}$ be an m-dimensional standard $\mathcal{F}_t$-adapted Brownian motion, and $\{k_t, t\geqslant 0\}$ a stationary $\mathcal{F}_t$-adapted Poisson point process with values in $\mathbb{U}$ and with characteristic measure n(cf. [7]). Let $N_k((0,t],\mathrm{d}u)$ be the counting measure of k_t, i.e., for $A \in \mathcal{U}$

$$N_k((0,t],A) := \#\{0 < s \leqslant t : k_s \in A\},$$

where # denotes the cardinality of a set. The compensator measure of N_k is given by

$$\tilde{N}_k((0,t],\mathrm{d}u) := N_k((0,t],\mathrm{d}u) - tn(\mathrm{d}u).$$

Fix a $\mathbb{U}_0 \in \mathcal{U}$ such that $n(\mathbb{U}-\mathbb{U}_0) < \infty$, and consider the following SDE with jumps in $\mathbb{R}^d$:

$$X_t(x) = x + \int_0^t b(X_s(x))\,\mathrm{d}s + \int_0^t \sigma(X_s(x))\,\mathrm{d}W_s$$

$$+\int_0^{t+}\int_{\mathbb{U}_0} f(X_{s-}(x),u)\,\tilde{N}_k(\mathrm{d}s,\mathrm{d}u)$$
$$+\int_0^{t+}\int_{\mathbb{U}-\mathbb{U}_0} g(X_{s-}(x),u)\,N_k(\mathrm{d}s,\mathrm{d}u), \quad t\geqslant 0, \tag{5}$$

where $b:\mathbb{R}^d\mapsto\mathbb{R}^d$, $\sigma:\mathbb{R}^d\mapsto\mathbb{R}^d\times\mathbb{R}^m$, $f,g:\mathbb{R}^d\times\mathbb{U}\mapsto\mathbb{R}^d$ satisfy the following assumptions:

($\mathbf{H}_b$) there exists a constant $C_b>0$ such that for $x,y\in\mathbb{R}^d$

$$|b(x)-b(y)|\leqslant C_b|x-y|\cdot\log(|x-y|^{-1}+e);$$

($\mathbf{H}_\sigma$) there exists a constant $C_\sigma>0$ such that for $x,y\in\mathbb{R}^d$

$$|\sigma(x)-\sigma(y)|^2\leqslant C_\sigma|x-y|^2\cdot\log(|x-y|^{-1}+e);$$

($\mathbf{H}_f$) for some $q>(2d)\vee 4$ and any $p\in[2,q]$, there exists a constant $C_p>0$ such that for $x,y\in\mathbb{R}^d$

$$\int_{\mathbb{U}_0}|f(x,u)-f(y,u)|^p\,n(\mathrm{d}u)\leqslant C_p|x-y|^p\cdot\log(|x-y|^{-1}+e),$$

and

$$\int_{\mathbb{U}_0}|f(x,u)|^p\,n(\mathrm{d}u)\leqslant C_p(1+|x|)^p.$$

($\mathbf{H}_g$) for $u\in\mathbb{U}-\mathbb{U}_0$, $x\mapsto g(x,u)\in\mathcal{C}(\mathbb{R}^d)$, where $\mathcal{C}(\mathbb{R}^d)$ stands for the total of continuous functions from $\mathbb{R}^d$ to $\mathbb{R}^d$.

Here, the second integral of the right hand side in Eq.(5) is taken in Itô's sense, and the definitions of the third and fourth integrals are referred to [7].

Under **($\mathbf{H}_b$)**, **($\mathbf{H}_\sigma$)**, **($\mathbf{H}_f$)** and **($\mathbf{H}_g$)**, it is well known that there exists a unique strong solution to Eq.(5)(cf. [10]). This solution will be denoted by $X_t(x)$. Set

$$\mathscr{F}_{\geqslant t}:=\sigma\{W_s,N_p((0,s],B);s\geqslant t,B\in\mathcal{U}\},$$
$$\mathscr{F}_{\leqslant t}:=\sigma\{W_s,N_p((0,s],B);s\leqslant t,B\in\mathcal{U}\},$$

for $t\geqslant 0$. And then

$$\theta_t^{-1}\mathscr{F}_{>0}\subset\mathscr{F}_{>t}$$

and

$$\theta_t^{-1}\mathscr{F}_{\leqslant s}\subset\mathscr{F}_{\leqslant t+s}, \quad s\geqslant 0.$$

Moreover, $X_t(x)$ is $\mathscr{F}_{\leqslant t}$-measurable and independent of $\theta_t^{-1}\mathscr{F}_{>0}$.

Definition 3.1. *A probability measure μ on $(\Omega\times\mathbb{R}^d,\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^d))$ is called a Markov measure if μ_ω satisfies*

$$\mathbb{E}(\mu.|\mathscr{F}_{<\infty})=\mathbb{E}(\mu.|\mathscr{F}_{=0}), \quad P.a.s..$$

For the Markov process $X_t(x)$, the transition probability is defined by

$$p_t(x,B):=P\big(X_t(x)\in B\big), \quad t>0, B\in\mathcal{B}(\mathbb{R}^d).$$

Definition 3.2. *A measure $\bar{\mu}$ on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ is a stationary measure for p_t or Eq.(5) if*

$$\int_{\mathbb{R}^d}p_t(x,B)\bar{\mu}(\mathrm{d}x)=\bar{\mu}(B), \quad \forall t>0, B\in\mathcal{B}(\mathbb{R}^d).$$

Theorem 3.3. *Set $\varphi(t,\omega)x := X_t(x)$, and then for an invariant Markov measure μ of the skew product flow Θ_t, the stationary measure $\bar{\mu}$ for p_t is given by*

$$\bar{\mu} = \mathbb{E}(\mu_\cdot|\mathscr{F}_{>0}).$$

Proof. By Definition 3.1

$$\mathbb{E}(\mu_\cdot|\mathscr{F}_{<\infty}) = \mathbb{E}(\mu_\cdot|\mathscr{F}_{=0}).$$

Set

$$\bar{\mu} := \mathbb{E}(\mu_\cdot|\mathscr{F}_{>0}),$$

and then it follows from independence of $\mathscr{F}_{>0}$ and $\mathscr{F}_{=0}$

$$\begin{aligned} \bar{\mu} &= \mathbb{E}\left[\mathbb{E}(\mu_\cdot|\mathscr{F}_{=0})|\mathscr{F}_{>0}\right] \\ &= \mathbb{E}\left[\mathbb{E}(\mu_\cdot|\mathscr{F}_{=0})\right] \\ &= \mathbb{E}(\mu_\cdot). \end{aligned}$$

Therefore ν is not random.

By (3), it holds that for $t > 0$

$$\begin{aligned} \mathbb{E}[\varphi(t,\cdot)\mu_\cdot|\theta_t^{-1}\mathscr{F}_{>0}] &= \mathbb{E}[\mathbb{E}(\varphi(t,\cdot)\mu_\cdot|\theta_t^{-1}\mathcal{F})|\theta_t^{-1}\mathscr{F}_{>0}] \\ &= \mathbb{E}[\mu_{\theta_t\cdot}|\theta_t^{-1}\mathscr{F}_{>0}] \\ &= \mathbb{E}(\mu_\cdot|\mathscr{F}_{>0})(\theta_t\omega) \\ &= \bar{\mu}. \end{aligned}$$

Besides,

$$\begin{aligned} \mathbb{E}[\varphi(t,\cdot)\mu_\cdot|\theta_t^{-1}\mathscr{F}_{>0}] &= \mathbb{E}[\mathbb{E}(\varphi(t,\cdot)\mu_\cdot|\mathscr{F}_{>0})|\theta_t^{-1}\mathscr{F}_{>0}] \\ &= \mathbb{E}[\varphi(t,\cdot)\bar{\mu}|\theta_t^{-1}\mathscr{F}_{>0}] \\ &= \mathbb{E}[\varphi(t,\cdot)\bar{\mu}]. \end{aligned}$$

Thus for $B \in \mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned} \bar{\mu}(B) &= \mathbb{E}[\varphi(t,\cdot)\bar{\mu}(B)] = \mathbb{E}[\bar{\mu}(\varphi(t,\cdot)^{-1}B)] \\ &= \int_\Omega P(\mathrm{d}\omega)\int_{\mathbb{R}^d} 1_B(\varphi(t,\omega)x)\bar{\mu}(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \bar{\mu}(\mathrm{d}x)\int_\Omega 1_B(\varphi(t,\omega)x)P(\mathrm{d}\omega) \\ &= \int_{\mathbb{R}^d} p_t(x,B)\bar{\mu}(\mathrm{d}x). \end{aligned}$$

By Definition 3.2 $\mathbb{E}(\mu_\cdot|\mathscr{F}_{>0})$ is a stationary measure for p_t. □

So, this theorem, together with [10, Theorem 1.1] and [5, Lemma 5.1], yields

Theorem 3.4. *If $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is replaced by its one point compactification $(\hat{\mathbb{R}}^d, \mathcal{B}(\hat{\mathbb{R}}^d))$, stationary measures for Eq.(5) on $(\hat{\mathbb{R}}^d, \mathcal{B}(\hat{\mathbb{R}}^d))$ exist.*

4. From Fokker-Planck equations to stationary measures

Consider the following equation

$$X_t(x) = x + \int_0^t b(X_s(x))\,\mathrm{d}s + \int_0^t \sigma(X_s(x))\,\mathrm{d}W_s + L_t,$$

where L_t is a rotation invariant α-stable process independent of W_t. Based on the lévy-Itô representation of L_t, the above equation can be rewritten as follows:

(i) for $1 \leqslant \alpha < 2$,

$$\begin{aligned} X_t(x) &= x + \int_0^t b(X_s(x))\,\mathrm{d}s + \int_0^t \sigma(X_s(x))\,\mathrm{d}W_s \\ &\quad + \int_0^t \int_{|u|\leqslant\delta} u\tilde{N}_k(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_{|u|>\delta} uN_k(\mathrm{d}s, \mathrm{d}u), \end{aligned} \tag{6}$$

where $k(t) := L_t - L_{t-}$ and $0 < \delta < 1$ satisfies that for some $q > 4d$

$$\frac{q\delta}{(1-\delta)^{q+1}} < 1;$$

(ii) for $0 < \alpha < 1$

$$\begin{aligned} X_t(x) &= x + \int_0^t b(X_s(x))\,\mathrm{d}s + \int_0^t \sigma(X_s(x))\,\mathrm{d}W_s \\ &\quad + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} uN_k(\mathrm{d}s, \mathrm{d}u). \end{aligned} \tag{7}$$

We study mainly Eq.(6). Eq.(7) can be dealt with similarly.

Under $(\mathbf{H}_b)$ and $(\mathbf{H}_\sigma)$, by [10, Theorem 1.3], for almost all $\omega \in \Omega$, $x \mapsto X_t(x,\omega)$ is a homeomorphism mapping on $\mathbb{R}^d$, where $X_t(x,\omega)$ is the solution of Eq.(6). Define

$$(p_t h)(x) := \mathbb{E}\big[h\big(X_t(x)\big)\big] = \int_{\mathbb{R}^d} h(y) p_t(x, \mathrm{d}y),$$

for $h \in \mathcal{C}_0(\mathbb{R}^d)$. Thus $p_t h \in \mathcal{C}_0(\mathbb{R}^d)$ by dominated convergence theorem. Let $\mathcal{M}_r(\mathbb{R}^d)$ be the set of all finite regular signed measures on $\mathcal{B}(\mathbb{R}^d)$. And then it is adjoint of $\mathcal{C}_0(\mathbb{R}^d)$(c.f. [14, Theorem 5.4.3, p.148].

Lemma 4.1. *The family of operators $\{p_t, t \geqslant 0\}$ defined above is a strongly continuous contraction semigroup on $\mathcal{C}_0(\mathbb{R}^d)$.*

Proof. For $t, s \geqslant 0$ and $h \in \mathcal{C}_0(\mathbb{R}^d)$, by C-K equation

$$\begin{aligned} (p_{t+s}h)(x) &= \int_{\mathbb{R}^d} h(y) p_{t+s}(x, \mathrm{d}y) = \int_{\mathbb{R}^d} h(y) \int_{\mathbb{R}^d} p_s(z, \mathrm{d}y) p_t(x, \mathrm{d}z) \\ &= \int_{\mathbb{R}^d} p_t(x, \mathrm{d}z) \int_{\mathbb{R}^d} h(y) p_s(z, \mathrm{d}y) = \int_{\mathbb{R}^d} p_t(x, \mathrm{d}z)(p_s h)(z) \\ &= (p_t(p_s h))(x). \end{aligned}$$

So, $p_{t+s} = p_t p_s$. Obviously $p_0 = I$.

Next, we prove strong continuity. For $h \in \mathcal{C}_0(\mathbb{R}^d)$, h is uniformly continuous on $\mathbb{R}^d$. And for $\forall \varepsilon > 0$, there exists an $\eta > 0$ such that $|h(x) - h(y)| < \varepsilon$ for $x, y \in \mathbb{R}^d$, $|x-y| < \eta$. For any $\lambda \in \mathcal{M}_r(\mathbb{R}^d)$

$$
\begin{aligned}
& \left| \int_{\mathbb{R}^d} (p_t h)(x)\lambda(\mathrm{d}x) - \int_{\mathbb{R}^d} h(x)\lambda(\mathrm{d}x) \right| \\
= & \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(y) p_t(x, \mathrm{d}y)\lambda(\mathrm{d}x) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x) p_t(x, \mathrm{d}y)\lambda(\mathrm{d}x) \right| \\
\leqslant & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |h(y) - h(x)| p_t(x, \mathrm{d}y)|\lambda|(\mathrm{d}x) \\
\leqslant & \int_{\mathbb{R}^d} \int_{|x-y|<\eta} |h(y) - h(x)| p_t(x, \mathrm{d}y)|\lambda|(\mathrm{d}x) \\
& + \int_{\mathbb{R}^d} \int_{|x-y|\geqslant\eta} |h(y) - h(x)| p_t(x, \mathrm{d}y)|\lambda|(\mathrm{d}x) \\
\leqslant & |\lambda|(\mathbb{R}^d)\varepsilon + 2\|h\| \int_{\mathbb{R}^d} P\{|X_t(x) - x| \geqslant \eta\}|\lambda|(\mathrm{d}x),
\end{aligned}
$$

where $|\lambda|$ stands for the variation measure of the signed measure λ. For $\int_{\mathbb{R}^d} P\{|X_t(x)-x| \geqslant \eta\}|\lambda|(\mathrm{d}x)$, by stochastical continuity of X_t and dominated convergence theorem, when t is small enough,

$$\int_{\mathbb{R}^d} P\{|X_t(x) - x| \geqslant \eta\}|\lambda|(\mathrm{d}x) < \varepsilon.$$

So,

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} (p_t h)(x)\lambda(\mathrm{d}x) = \int_{\mathbb{R}^d} h(x)\lambda(\mathrm{d}x).$$

That is to say, $p_t h$ converges weakly to h. By [16, Theorem, p.233], $p_t h$ converges strongly to h.

Finally, by [9, Definition 2.1, p.4], $\{p_t, t \geqslant 0\}$ is a strongly continuous contraction semigroup on $\mathcal{C}_0(\mathbb{R}^d)$. □

Let $\mathcal{L}$ be the infinitesimal generator of $\{p_t, t \geqslant 0\}$.

Lemma 4.2. *For $h \in \mathcal{C}_c^2(\mathbb{R}^d)$,*

$$(\mathcal{L}h)(x) = \langle \partial_x h(x), b(x) \rangle + \frac{1}{2} \left(\frac{\partial^2}{\partial x_i \partial x_j} h(x) \right) \sigma_{ij}(x) + (\mathcal{L}_\alpha h)(x).$$

Proof. By the Itô formula, one can obtain for $h \in \mathcal{C}_c^2(\mathbb{R}^d)$

$$
\begin{aligned}
h(X_t) - h(x) \quad = \quad & \int_0^t \langle \partial_y h(X_s), b(X_s) \rangle \mathrm{d}s + \int_0^t \langle \partial_y h(X_s), \sigma(X_s) \mathrm{d}W_s \rangle \\
& + \int_0^t \int_{|u|\leqslant\delta} (h(X_s + u) - h(X_s))\, \tilde{N}_p(\mathrm{d}s, \mathrm{d}u) \\
& + \int_0^t \int_{|u|>\delta} (h(X_s + u) - h(X_s))\, N_p(\mathrm{d}s, \mathrm{d}u)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^t\left(\frac{\partial^2}{\partial y_i\partial y_j}h(X_s)\right)\sigma_{ij}(X_s)\mathrm{d}s\\
&+\int_0^t\int_{|u|\leqslant\delta}\big(h(X_s+u)-h(X_s)\\
&\qquad -\langle\partial_y h(X_s),u\rangle\big)\frac{C_{d,\alpha}}{|u|^{d+\alpha}}\mathrm{d}u\mathrm{d}s.
\end{aligned}
$$

Taking expectation on two sides, we get

$$
\begin{aligned}
(p_th)(x)-h(x) &= \int_0^t\mathbb{E}\left[\langle\partial_y h(X_s),b(X_s)\rangle\right]\mathrm{d}s+\frac{1}{2}\int_0^t\mathbb{E}\left[\left(\frac{\partial^2}{\partial y_i\partial y_j}h(X_s)\right)\sigma_{ij}(X_s)\right]\mathrm{d}s\\
&\quad+\int_0^t\mathbb{E}\bigg[\int_{\mathbb{R}^d\setminus\{0\}}\big(h(X_s+u)-h(X_s)\\
&\qquad-\langle\partial_y h(X_s),u\rangle\big)\frac{C_{d,\alpha}}{|u|^{d+\alpha}}\mathrm{d}u\bigg]\mathrm{d}s\\
&= \int_0^t\int_{\mathbb{R}^d}\langle\partial_y h(y),b(y)\rangle p_s(x,\mathrm{d}y)\mathrm{d}s\\
&\quad+\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\left(\frac{\partial^2}{\partial y_i\partial y_j}h(y)\right)\sigma_{ij}(y)p_s(x,\mathrm{d}y)\mathrm{d}s\\
&\quad+\int_0^t\int_{\mathbb{R}^d}\bigg[\int_{\mathbb{R}^d\setminus\{0\}}(h(y+u)-h(y)\\
&\qquad-\langle\partial_y h(y),u\rangle)\frac{C_{d,\alpha}}{|u|^{d+\alpha}}\mathrm{d}u\bigg]p_s(x,\mathrm{d}y)\mathrm{d}s.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\lim_{t\downarrow 0}\frac{1}{t}\left((p_th)(x)-h(x)\right) &= \langle\partial_x h(x),b(x)\rangle+\frac{1}{2}\left(\frac{\partial^2}{\partial x_i\partial x_j}h(x)\right)\sigma_{ij}(x)\\
&\quad+\int_{\mathbb{R}^d\setminus\{0\}}\left(h(x+u)-h(x)-\langle\partial_x h(x),u\rangle\right)\frac{C_{d,\alpha}}{|u|^{d+\alpha}}\mathrm{d}u.
\end{aligned}
$$

By [11, Lemma 31.7, p.209],

$$
\begin{aligned}
(\mathcal{L}h)(x) &= \langle\partial_x h(x),b(x)\rangle+\frac{1}{2}\left(\frac{\partial^2}{\partial x_i\partial x_j}h(x)\right)\sigma_{ij}(x)\\
&\quad+\int_{\mathbb{R}^d\setminus\{0\}}\left(h(x+u)-h(x)-\langle\partial_x h(x),u\rangle\right)\frac{C_{d,\alpha}}{|u|^{d+\alpha}}\mathrm{d}u\\
&= \langle\partial_x h(x),b(x)\rangle+\frac{1}{2}\left(\frac{\partial^2}{\partial x_i\partial x_j}h(x)\right)\sigma_{ij}(x)+(\mathcal{L}_\alpha h)(x).
\end{aligned}
$$

$\square$

To get the main result, we make the following assumptions.

($\mathbf{H}_p$) For all $t > 0$ and $x \in \mathbb{R}^d$, the transition probability $p_t(x, \mathrm{d}y)$ admits a density $\rho_t(x,y)$ and functions

$$\begin{array}{rcl} (t,y) & \mapsto & \frac{\partial}{\partial t}\rho_t(x,y), \\ (t,y) & \mapsto & \frac{\partial}{\partial y_j}\big(b_j(y)\rho_t(x,y)\big), j = 1,2,\ldots,d, \\ (t,y) & \mapsto & \frac{\partial^2}{\partial y_i \partial y_j}\big(\sigma_{ij}(y)\rho_t(x,y)\big), i,j = 1,2,\ldots,d, \\ (t,y) & \mapsto & \big(\mathcal{L}_\alpha \rho_t(x,\cdot)\big)(y), \end{array}$$

exist and are continuous on $\mathbb{R}_+ \times \mathbb{R}^d$.

Under ($\mathbf{H}_p$), the distribution of $X_t(x)$ has a density $\rho_X(t,y) = \rho_t(x,y)$ for $t > 0$.

Lemma 4.3. *$\rho_X(t,y)$ satisfies the following Fokker-Planck equation*

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t}u(t,y) = (\mathcal{L}^* u(t,\cdot))(y), \\ \lim\limits_{t\downarrow 0} u(t,y)\mathrm{d}y = \delta_x(\mathrm{d}y), \end{array} \right. \tag{8}$$

where $\mathcal{L}^*$ is adjoint of $\mathcal{L}$, i.e.

$$(\mathcal{L}^*\phi)(y) := -\frac{\partial}{\partial y_j}\big(b_j(y)\phi(y)\big) + \frac{1}{2}\frac{\partial^2}{\partial y_i \partial y_j}\big(\sigma_{ij}(y)\phi(y)\big) + (\mathcal{L}_\alpha\phi)(y)$$

for $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.

Proof. By similar deduction to that of Lemma 4.2, we get for $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and any $r > 0$

$$\begin{array}{rl} & \int_{\mathbb{R}^d} \psi(y)\rho_X(t+r,y)\mathrm{d}y - \int_{\mathbb{R}^d} \psi(y)\rho_X(t,y)\mathrm{d}y \\ = & \int_t^{t+r}\int_{\mathbb{R}^d} \langle \partial_y\psi(y), b(y)\rangle \rho_X(s,y)\mathrm{d}y\mathrm{d}s \\ & +\frac{1}{2}\int_t^{t+r}\int_{\mathbb{R}^d} \left(\frac{\partial^2}{\partial y_i\partial y_j}\psi(y)\right)\sigma_{ij}(y)\rho_X(s,y)\mathrm{d}y\mathrm{d}s \\ & +\int_t^{t+r}\int_{\mathbb{R}^d} (\mathcal{L}_\alpha\psi)(y)\rho_X(s,y)\mathrm{d}y\mathrm{d}s. \end{array}$$

Divided by r and Letting $r \downarrow 0$, it follows from dominated convergence theorem and integration by parts that

$$\begin{array}{rcl} \int_{\mathbb{R}^d} \psi(y)\frac{\partial}{\partial t}\rho_X(t,y)\mathrm{d}y & = & \int_{\mathbb{R}^d} \langle \partial_y\psi(y), b(y)\rangle\rho_X(t,y)\mathrm{d}y \\ & & +\frac{1}{2}\int_{\mathbb{R}^d}\left(\frac{\partial^2}{\partial y_i\partial y_j}\psi(y)\right)\sigma_{ij}(y)\rho_X(t,y)\mathrm{d}y \\ & & +\int_{\mathbb{R}^d}(\mathcal{L}_\alpha\psi)(y)\rho_X(t,y)\mathrm{d}y \\ & = & -\int_{\mathbb{R}^d}\psi(y)\frac{\partial}{\partial y_j}\big(b_j(y)\rho_X(t,y)\big)\mathrm{d}y \end{array}$$

$$+\frac{1}{2}\int_{\mathbb{R}^d}\psi(y)\frac{\partial^2}{\partial y_i\partial y_j}\big(\sigma_{ij}(y)\rho_X(t,y)\big)\mathrm{d}y$$
$$+\int_{\mathbb{R}^d}\psi(y)(\mathcal{L}_\alpha\rho_X(t,\cdot))(y)\mathrm{d}y.$$

Thus

$$\begin{aligned}\frac{\partial}{\partial t}\rho_X(t,y) &= -\frac{\partial}{\partial y_j}\big(b_j(y)\rho_X(t,y)\big)+\frac{1}{2}\frac{\partial^2}{\partial y_i\partial y_j}\big(\sigma_{ij}(y)\rho_X(t,y)\big)+(\mathcal{L}_\alpha\rho_X(t,\cdot))(y)\\ &= (\mathcal{L}^*\rho_X(t,\cdot))(y).\end{aligned}$$

□

Note that Schertzer et. al. [12] also derived the Fokker-Planck equation in this context. The main result in this section is the following theorem.

Theorem 4.4. *If $\rho(y)\in\mathbb{H}^{2,2}(\mathbb{R}^d)$ satisfies the following equation*

$$\mathcal{L}^*\rho=0$$

and

$$\rho(y)\geqslant 0,\forall y\in\mathbb{R}^d \qquad \text{and} \qquad \int_{\mathbb{R}^d}\rho(y)\mathrm{d}y=1.$$

Then $\bar{\mu}(\mathrm{d}y):=\rho(y)\mathrm{d}y$ is a stationary measure for p_t.

Proof. By Lemma 4.1 and 4.2, $\{p_t,t\geqslant 0\}$ is a strongly continuous semigroup on $\mathcal{C}_0(\mathbb{R}^d)$ with the infinitesimal generator $\mathcal{L}$. Let p_t^* be adjoint of p_t. By Theorem 2.5, $\mathbb{H}^{2,2}(\mathbb{R}^d)$ is the closure of $\mathcal{D}(\mathcal{L}^*)$ in $\mathcal{M}_r(\mathbb{R}^d)$. [9, Theorem 10.4, p.41] admits us to get that the restriction p_t^+ of p_t^* to $\mathbb{H}^{2,2}(\mathbb{R}^d)$ is a strongly continuous semigroup on $\mathcal{M}_r(\mathbb{R}^d)$. Moreover, the infinitesimal generator $\mathcal{L}^+$ of p_t^+ is the part of $\mathcal{L}^*$ in $\mathbb{H}^{2,2}(\mathbb{R}^d)$, i.e. $\mathcal{D}(\mathcal{L}^+)=\{h\in\mathcal{D}(\mathcal{L}^*)\cap\mathbb{H}^{2,2}(\mathbb{R}^d),\mathcal{L}^*h\in\mathbb{H}^{2,2}(\mathbb{R}^d)\}$ and $\mathcal{L}^+h=\mathcal{L}^*h$ for $h\in\mathcal{D}(\mathcal{L}^+)$. Thus, Eq.(8) has a unique solution in $\mathbb{H}^{2,2}(\mathbb{R}^d)$ by [9, Theorem 1.3, p.102].

Next, since $\rho(y)$ satisfies

$$\mathcal{L}^*\rho=0$$

and

$$\rho(y)\geqslant 0,\forall y\in\mathbb{R}^d \qquad \text{and} \qquad \int_{\mathbb{R}^d}\rho(y)\mathrm{d}y=1,$$

$\rho(y)$ solves Eq.(8) and is a density function. So,

$$\rho_X(t,y)=\rho(y), \quad t>0.$$

For $t>0$ and $B\in\mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d}p_t(x,B)\bar{\mu}(\mathrm{d}x) &= \int_{\mathbb{R}^d}p_t(x,B)\rho(x)\mathrm{d}x=\int_{\mathbb{R}^d}p_t(x,B)\rho_X(s,x)\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\int_B\rho_t(x,y)\rho_X(s,x)\mathrm{d}y\mathrm{d}x\\ &= \int_B\int_{\mathbb{R}^d}\rho_t(x,y)\rho_X(s,x)\mathrm{d}x\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= \int_B \rho_X(s+t,y)\mathrm{d}y \\
&= \int_B \rho(y)\mathrm{d}y = \bar{\mu}(B).
\end{aligned}
$$

By Definition 3.2, $\bar{\mu}(\mathrm{d}y)$ is a stationary measure for p_t. $\square$

Remark 4.5. *If $b(x) = -x$ and $\sigma(x) = 0$, the above theorem is* [1, Proposition 3.2(ii)]. *Moreover,*

$$
\hat{\rho}(u) = \exp\{-\frac{1}{\alpha}C|u|^{\alpha}\}, \quad u \in \mathbb{R}^d,
$$

where C is the same constant as one in Definition 2.4.

5. From probability density functions to stationary measures

In the section we study Eq.(5) with $g = 0$. We further make the following assumptions.

($\mathbf{H}^1_{b,\sigma,f}$) b and σ are $(4d+6)$-times differentiable with bounded derivatives of all order between 1 and $4d+6$. Besides, $f(\cdot,u)$ is $(4d+6)$-times differentiable, and

$$
\begin{aligned}
f(0,\cdot) &\in \bigcap_{2\leqslant q<\infty} L^q(\mathbb{U}_0, n) \\
\sup_x |\partial_x^r f(x,\cdot)| &\in \bigcap_{2\leqslant q<\infty} L^q(\mathbb{U}_0, n), \quad 1 \leqslant r \leqslant 4d+6,
\end{aligned}
$$

where the space $(\mathbb{U}_0, n)$ is equipped with a norm and $\partial_x^r f(x,\cdot)$ stands for r order partial derivative of $f(x,\cdot)$ with respect to x.

($\mathbf{H}^2_{b,\sigma,f}$) There exist three constants $\varepsilon > 0$, $\delta \geqslant 0$ and $C > 0$ such that for all $x, y \in \mathbb{R}^d$

$$
\langle y, \sigma(x)\sigma^T(x)y\rangle \geqslant |y|^2 \frac{\varepsilon}{1+|x|^{\delta}}
$$

and

$$
|\det\{I + r\partial_x f(x,u)\}| \geqslant C
$$

for all $r \in [0,1]$.

Under ($\mathbf{H}^1_{b,\sigma,f}$) and ($\mathbf{H}^2_{b,\sigma,f}$), by [4, Theorem 2-29, p.15], Eq.(5) has a unique solution denoted by X_t. Moreover, the transition probability $p_t(x,\mathrm{d}y)$ has a density $\rho_t(x,y)$ and $(t,x,y) \mapsto \rho_t(x,y)$ is continuous. Thus, the distribution of X_t has a density $\rho_X(t,y) = \rho_t(x,y)$.

Theorem 5.1. *Suppose that $\lim_{t\to\infty} \rho_X(t,y) = \rho(y)$, where $\rho(y)$ satisfies*

$$
\rho(y) \geqslant 0, \forall y \in \mathbb{R}^d \qquad \text{and} \qquad \int_{\mathbb{R}^d} \rho(y)\mathrm{d}y = 1.
$$

Then $\bar{\mu}(\mathrm{d}y) := \rho(y)\mathrm{d}y$ is a stationary measure for p_t.

Proof. For $t > 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$,

$$
\begin{aligned}
\int_{\mathbb{R}^d} p_t(x,B)\bar{\mu}(\mathrm{d}x) &= \int_{\mathbb{R}^d} p_t(x,B)\rho(x)\mathrm{d}x = \int_{\mathbb{R}^d} p_t(x,B) \lim_{s\to\infty} \rho_X(s,x)\mathrm{d}x \\
&= \lim_{s\to\infty} \int_{\mathbb{R}^d} p_t(x,B)\rho_X(s,x)\mathrm{d}x = \lim_{s\to\infty} \int_{\mathbb{R}^d}\int_B \rho_t(x,y)\rho_X(s,x)\mathrm{d}y\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{s\to\infty}\int_B\int_{\mathbb{R}^d}\rho_t(x,y)\rho_X(s,x)\mathrm{d}x\mathrm{d}y\\
&= \lim_{s\to\infty}\int_B \rho_X(s+t,y)\mathrm{d}y\\
&= \bar{\mu}(B).
\end{aligned}
$$

By Definition 3.2, $\bar{\mu}(\mathrm{d}y)$ is a stationary measure for p_t. □

Remark 5.2. *By the above theorem, we see that if a limiting distribution exists, it must be a stationary measure. This theorem also has a corresponding version in the theory of Markov chains (c.f.* [8, p.237]*).*

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